ISRAEL JOURNAL OF MATHEMATICS, Vol. 48, Nos. 2-3, 1984

SPECTRAL THEORY OF GROUP REPRESENTATIONS AND THEIR NONSTANDARD HULL

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Dedicated to Prof. Dr. H. H. Schaefer on the occasion of his 60th birthday

ABSTRACT
We construct the nonstandard hull of a not necessarily bounded strongly continuous representation U of the locally compact semigroup S on a Banach space E. Then we apply our results to the theory of the spectrum $\sigma(U)$ of U, mainly in cases where S is an abelian group, e.g. $S = \mathbf{R}$. First of all we obtain generalizations to the unbounded case of results known for the bounded one. Secondly we introduce the notion of the Riesz part $\mathrm{R}\,\sigma(U)$ of $\sigma(U)$ and characterize those representations satisfying $\sigma(U) = \mathrm{R}\,\sigma(U)$. We illustrate the theory developed so far by applications to representations on Banach lattices.

Introduction

Let U be a strongly continuous representation of the locally compact abelian group G on a Banach space E. Arveson [2] introduced the notion of the spectrum in case U is bounded, and independently of [2] Lyubich [18] defined the spectrum in a different way for a general representation U. Both notions agree if U is bounded ([6], sect. 6; [24], 8.1). Moreover, in [1] the spectral calculus of Arveson was generalized to non-quasianalytic representations which were considered for the first time in [8] (cf. [9]).

In the present paper our main aim is to investigate further the spectrum $\sigma(U)$ of a representation U. In particular we introduce the notion of the Riesz part $\mathrm{R}\,\sigma(U)$ of $\sigma(U)$ and characterize completely those non-quasianalytic representations for which $\sigma(U)$ equals $\mathrm{R}\,\sigma(U)$. In the last section of this paper we apply the general results to representations of the group G by positive operators on Banach lattices.

Received March 3, 1983 and in revised form March 19, 1984

As an important tool we introduce the nonstandard hull $(\tilde{E}, \tilde{U})$ for a representation U of the locally compact semigroup S on a Banach space E thus carrying on ideas implicitly contained already in [7, 18, 19]. The basic two properties of $\tilde{U}$ are first of all that the two associated representations U^1, $\tilde{U}^1$, respectively, of the measure algebra are compatible (see 2.2 for a precise formulation), and secondly that in case S is a group $\sigma(U)$ equals $\sigma(\tilde{U})$ and this latter set is equal to the set $P\sigma(\tilde{U})$ of all "eigenvalues" of $\tilde{U}$. The first-mentioned property is not shared in general by weaker "standard" extensions of U.

We now summarize briefly the particular sections: In section 1 we introduce the nonstandard hull $(\tilde{E}, \tilde{U})$, whereas the second section is devoted to the associated representations of the measure algebra. The main results concerning the relationship between $\sigma(U)$ and $\sigma(\tilde{U})$ are contained in section 3. In the fourth section we give first some easy applications of the theory developed so far. The fifth section contains the main application to the Riesz part of the spectrum introduced there, whereas section 6 is devoted to representations on Banach lattices.

In order to make the paper instructive for mathematicians not familiar with nonstandard analysis we label most standard results and definitions by the letter S.

I wish to thank Prof. W. Luxemburg for valuable discussions after the talk I gave about this subject on the conference on nonstandard analysis at Oberwolfach in June 1981. In addition I thank Dr. W. Arendt (Tübingen) for helpful conversations as well as for the preprint [3].

Finally I thank the referee for having pointed out some misprints and ununderstandable formulations, in particular in the proof of 4.4.

§1. The nonstandard hull of a strongly continuous representation of a locally compact semigroup

1.1. For notions of nonstandard analysis not explained here we refer to [17, 26]. The enlargements we use are always tacitly assumed to be polysaturated. The standard super structure $\mathcal{M}$ under consideration is assumed to contain all standard entities in question (like $\mathbf{C}$, the Banach space E, etc.). The counterpart in the enlargement $^*\mathcal{M}$ of a standard entity A is sometimes denoted by $^*A$. But if no confusion is possible we omit the star.

For a topological space X, $\mathrm{ns}(^*X)$ denotes the set of all near standard points. If x, $y \in \mathrm{ns}(^*X)$ and both have the same standard part then we write $x \approx y$.

More generally if the topology is induced by a uniform structure $\mathcal{N}$ we extend $\approx$ to all of $^*X$ by setting $x \approx y$ iff (x, y) is in the monad of $\mathcal{N}$.

For a normed space $(E, \|.\|)$, $\operatorname{fin}(^*E)$ is the set of all finite elements of $^*E$, $\inf(^*E)$ denotes the infinitesimals, $\hat{E} := \operatorname{fin}(^*E)/\inf(^*E)$ is the nonstandard hull of E. The canonical embedding of the space $\mathcal{L}(E)$ of all bounded linear operators on E into $\mathcal{L}(\hat{E})$ is given by $T \to \hat{T}$ $(\hat{T}\hat{x} := (^*Tx)^{\wedge})$.

1.2. In the following let $(S, .)$ denote a locally compact semigroup with unit e (we always assume the multiplication to be jointly continuous). A strongly continuous representation U of S on the Banach space E is a continuous homomorphism of S into $\mathcal{L}(E)$, this space being equipped with the strong operator topology. Moreover we assume always $U_e = I$ (identity on E). U is called uniformly continuous (by abuse of language) if it is continuous with respect to the operator norm on $\mathcal{L}(E)$.

Now let U be a fixed strongly continuous representation of S on E. Define $E_S := \{x \in \operatorname{fin}(^*E) : U_s x \approx U_t x \text{ for } s, t \in \operatorname{ns}(^*S),\ s \approx t\}$.

The essential properties of E_S are collected in the following lemma.

1.3. LEMMA. (a) *E_S is a linear subspace of* $\operatorname{fin}(^*E)$, *which is invariant under U_t for all $t \in \operatorname{ns}(^*S)$. Moreover for each fixed $x \in E_S$ the mapping $t \to U_t x$ is s-continuous from* $\operatorname{ns}(^*S)$ *into* $^*E$.

(b) *E_S contains* $\inf(^*E)$, *and this latter space is also invariant under $\{U_t : t \in \operatorname{ns}(^*S)\}$.*

(c) *If $T \in \mathcal{L}(E)$ commutes with all U_t $(t \in S)$ then $^*T(E_S) \subset E_S$.*

(d) *If S is a locally compact group then $E_S = \{x \in \operatorname{fin}(^*E) : U_t x \approx x$ for all $t \approx e\}$.*

(e) *Let E be a Banach lattice (a C^*-algebra, resp.) and assume that $U(S)$ consists of lattice isomorphisms (*-automorphisms, resp.). Then E_S is a vector sublattice (a *-subalgebra) of* $\operatorname{fin}(^*E)$.

PROOF. Almost all is obvious, e.g., in order to prove $\inf(^*E) \subset E_S$ one uses the fact that $t \to \|U_t\|$ is bounded on compact subsets according to the uniform boundedness principle.

We are now able to define the nonstandard hull of the representation U. Denote the quotient mapping from $\operatorname{fin}(^*E)$ onto $\hat{E}$ by Q.

1.4. PROPOSITION. (a) *$\hat{E}_S := E_S/\inf(^*E)$ is a closed linear subspace of $\hat{E}$ containing E. Moreover the (not necessarily continuous) representation $\hat{U}$ of S on $\hat{E}$, given by $\hat{U}_t Q = Q^*U_t$ is well-defined. E_S is left invariant under $\hat{U}(S)$, and*

$\tilde{U}: S \to \mathcal{L}(\tilde{E}_S)$, given by $\tilde{U}_t = \hat{U}_{t|\tilde{E}_S}$ (restriction to $\tilde{E}_S$) is strongly continuous and satisfies $\tilde{U}_{t|E} = U_t$.

(b) *Let T be in the commutant $U(S)'$ of $U(S)$ in $\mathcal{L}(E)$. Then $\hat{T}(\tilde{E}_S) \subset \tilde{E}_S$ and $T \to \tilde{T} := \hat{T}_{|\tilde{E}_S}$ is an isometric embedding of $U(S)'$ into $\mathcal{L}(\tilde{E}_S)$.*

(c) *If E is a Banach lattice (a C^*-algebra) and if $U(S)$ consists of lattice homomorphisms (*-homomorphisms, respectively) then $\tilde{E}_S$ is a Banach lattice (a C^*-algebra) and $\tilde{U}(S)$ consists of lattice homomorphisms (*-homomorphisms).*

PROOF. In view of 1.3 we only have to prove

(i) $\tilde{E}_S$ is closed in $\hat{E}$.

(ii) $\tilde{U}$ is strongly continuous.

ad (i): Let $(\hat{x}_n)$ be a sequence in $\tilde{E}_S$ converging to $\hat{y} \in \hat{E}$. Choose $x_n \in \hat{x}_n$, $y \in \hat{y}$. By 1.3(b) $x_n \in E_S$ hence for $s, t \in \mathrm{ns}({}^*S)$ satisfying $u := \mathrm{st}(s) = \mathrm{st}(t)$ we have $U_s x_n \approx U_t x_n$. Choose a standard compact neighbourhood of u. There U is bounded by M, say. Moreover to each $\varepsilon > 0$ there exists n standard such that $\| x_n - y \| < \varepsilon / 2M$. Thus

$$\| U_s y - U_t y \| < \| U_s y - U_s x_n \| + \| U_s x_n - U_t x_n \| + \| U_t x_n - U_t y \| < \varepsilon.$$

This proves $\hat{y} \in \tilde{E}_S$.

ad (ii): Let $\hat{x} = Qx \in \tilde{E}_S$ be arbitrary. For $\varepsilon > 0$ standard and $t \in S$ standard the neighbourhood monad $\mu(t)$ is contained in the internal set $\{s \in {}^*S : \| U_s x - U_t x \| < \varepsilon\} =: C$. Hence by Cauchy's principle ([26], 7.6.4) there is a standard neighbourhood V of t such that ${}^*V \subset C$. Applying Q we obtain the assertion.

1.5. DEFINITION. The representation $\tilde{U}$ on $\tilde{E}_S$ is called the *nonstandard hull* of U.

The question under which hypotheses $\tilde{E}_S = \hat{E}$ may hold will be answered completely as follows:

1.6. PROPOSITION. *The following assertions are equivalent:*

(a) $\tilde{E}_S = \hat{E}$.

(b) $E_S = \mathrm{fin}({}^*E)$.

(c) *U is uniformly continuous.*

(d) *The representation $t \to \hat{U}_t$ on $\hat{E}$ is uniformly continuous.*

PROOF. In view of 1.3(b) all is obvious except (b) $\Rightarrow$ (c): Let B denote the unit ball in E, and let $\varepsilon > 0$ be a standard given number. Then for $t \in S$ standard the neighbourhood monad of t is contained in

$$D := \{s \in {}^*S : \forall x[x \in {}^*B \Rightarrow \| U_s x - U_t x \| < \varepsilon]\}.$$

By Cauchy's principle ([26], 7.6.4) there exists a standard neighbourhood V of t such that $^*V \subset D$. The assertion follows.

One may conjecture that $\tilde{E}_S$ is equal to $F := \{\hat{x} \in \hat{E} : t \rightarrow \hat{U}_t\hat{x}$ is continuous$\}$. But this is not true in general as the following example will show. The reason for the choice of $\tilde{E}_S$ instead of F will become clear from 2.3.

1.7. Example. Let $E = C(\Pi)$ be the space of all continuous complex-valued functions on the group $\Pi = \{z \in \mathbf{C} : |z| = 1\}$. Define U by $(U_z f)(v) = f(zv)$. By 26.14 in [16] to each finite subset M of Π and $\varepsilon > 0$ there exists $k \in \mathbf{Z}$, $k \neq 0$ satisfying $|z^k - 1| < \varepsilon$ for all $z \in M$. Thus to a $*$-finite set M of $^*\Pi$ containing Π and to $\varepsilon \approx 0$ there exists $k \in {}^*\mathbf{Z} \setminus \mathbf{Z}$ satisfying $|z^k - 1| < \varepsilon$ for all $z \in M$, in particular for all $z \in \Pi$. Hence $f : v \rightarrow v^k$ satisfies $\hat{U}_z \hat{f} = \hat{f}$ for all $z \in \Pi$, so $\hat{f} \in F$. On the other hand $f \notin E_\Pi$ by 1.3(d). So $F \neq \tilde{E}_\Pi$ by 1.3(b).

§2. The representation of measures

Assumption: All semigroups under consideration are locally compact and possess a unit, all representations are strongly continuous and map the unit onto the identity operators. All measures are Radon-measures. E always denotes a Banach space.

2.1. Let U be a representation of the semigroup S on E. Then the function $p : t \rightarrow \max(1, \| U_t \|)$ is lower semicontinuous, bounded on compact subsets and satisfies $p(st) \leqq p(s)p(t)$. Denote by $M(S, U)$ the space of all measures on S satisfying $\| \mu \| := \int p(t) d|\mu|(t) < \infty$. Then equipped with this norm and with convolution as multiplication $M(S, U)$ is a Banach algebra. The representation U can be extended to a contractive representation U^1 of $M(S, U)$ into $\mathscr{L}(E)$ by $U^1_\mu x = \int U_t x d\mu(t)$ (the integral in the weak topology). Let $\tilde{U}$ be the nonstandard hull of U on $\tilde{E}_S$. The next result states $\tilde{U}^1_\mu = (U^1_\mu)^\wedge | \tilde{E}_S$ (restriction of $U^{1\wedge}_\mu$ onto $\tilde{E}_S$). In more details:

2.2. Theorem. *Let $\mu \in M(S, U)$ be a standard measure and let $x \in E_S$ be arbitrary. Then $U^1_\mu(x) = \int U_t x d\mu(t)$ holds (where the right-hand side denotes the weak integral transferred to $^*\mathcal{M}$). Moreover this element is in E_S and we have $\hat{U}^1_\mu \hat{x} = \int \tilde{U}_t \hat{x} d\mu(t)$ $(= (\tilde{U})^1_\mu \hat{x})$.*

Proof. The first equality follows from the transfer principle and the definition of U^1_μ. Concerning the other statements let us first assume that μ has compact support K. By [26], 10.3.1 the nonstandard hull $(E')^\wedge$ of the dual E' of E separates the points of $\hat{E}$, hence it is enough to show $\langle \hat{y}, \hat{y}' \rangle = \langle \hat{z}, \hat{y}' \rangle$ for all

$y' \in \operatorname{fin}({}^*E')$, where y denotes ${}^*U^1_\mu x$ and $\hat{z} := \int \tilde{U}_t\hat{x}d\mu(t)$. Since $x \in E_S$ the function $t \to \langle U_t x, y'\rangle$ is s-continuous on ${}^*K$. Its standard part is $\langle \tilde{U}_t\hat{x}, \hat{y}'\rangle =: g(t)$ $(t \in K)$ and by [26], 8.4.41 ${}^*g(s) \approx \langle U_s x, y'\rangle$ holds on ${}^*K$. Thus

$$\langle \hat{y}, \hat{y}'\rangle = \operatorname{st}(\langle y, y'\rangle) = \operatorname{st}\left(\int \langle U_t x, y'\rangle d\mu(t)\right)$$
$$\approx \int_K \operatorname{st}(\langle U_t x, y'\rangle)d\mu(t) = \int_K \langle \hat{U}_t\hat{x}, \hat{y}'\rangle d\mu(t) = \langle \hat{z}, \hat{y}'\rangle.$$

This implies our assertion in the case under consideration.

Now assume the general case. To every standard $\varepsilon > 0$ there exists a compact standard set K such that $\|\mu - \mu_K\| < \varepsilon/2$ (μ_K: restriction of μ to K). Then for $x \in E_S$ we have

$$\left\|\int U_t x d\mu(t) - \int_{{}^*K} U_t x d\mu(t)\right\| < \frac{\varepsilon}{2}\|x\|, \qquad \left\|\int \tilde{U}_t\hat{x} d\mu(t) - \int_K \tilde{U}_t\hat{x} d\mu(t)\right\| < \frac{\varepsilon}{2}\|\hat{x}\|.$$

The theorem is proved.

We finish this section with an additional important theorem in case S is a locally compact group G with fixed left Haar measure dt. Then

$$L^1(G, U) := \left\{f \in L^1(G) : \int |f(t)|\, p(t)dt < \infty\right\}$$

is (identifiable with) a closed convolution ideal of $M(G, U)$. Moreover $L^1(G, U)$ possesses a bounded approximate identity. Thus $\{U^1_f x : x \in E, f \in L^1(G, U)\}$ is dense in E $(U^1_f x := \int f(t)U_t x dt)$.

2.3. THEOREM. *Let U be a representation of the group G on a Banach space E, and let $\tilde{U}$ denote the nonstandard hull of E on $\tilde{E}_G$. Then for all $f \in L^1(G, U)$, $(U^1_f)^\wedge$ maps the whole nonstandard hull $\hat{E}$ of E into the subspace $\tilde{E}_G$. Moreover $\{U^1_f\hat{x} : \hat{x} \in \tilde{E}_G, f \in L^1(G, U)\}$ is dense in $\tilde{E}_G$.*

PROOF. The second assertion is clear by the preceding paragraph. Concerning the first statement let $f \in L^1(G, U)$ be standard, and let $t \in {}^*G$ satisfy $t \approx e$. Then since the representation V on $L^1(G, U)$ given by $V_s f = \varepsilon_s * f$ is strongly continuous we get $\|\varepsilon_t * f - f\| \approx 0$. Hence for all $x \in \operatorname{fin}({}^*E)$, $\|U_t U^1_f x - U^1_f x\| \approx 0$ holds, thus $U^1_f x \in E_G$ by 1.3(d), and the theorem is proved.

§3. Spectral theory of group representations

Assumption: In addition to the general assumption made in section 2 we consider from now on only locally compact abelian groups, denoted by G, with fixed Haar measure dt. The group of all unbounded continuous characters of G

(into $\mathbf{C}\setminus\{0\}$) is denoted by $G^{\times}_{\infty}$. It is equipped with the compact-open topology. Its subgroup of all bounded characters is denoted by $G^{\times}$ and is called the dual group. The Fourier-transform of a measure μ is given by $\tilde{\mu}(\chi) = \int \chi(t)d\mu(t)$. It is defined for all $\chi \in G^{\times}_{\infty}$ for which $\int |\chi(t)| d|\mu|(t) < \infty$ holds.

3.1. (S) Definition. A representation U of G on E is called *bounded* if $U(G)$ is bounded in $\mathscr{L}(E)$. U is called *non-quasianalytic* ([12]) (nqa for short) if for all $t \in G$, $\Sigma_1^{\infty}(1+k^2)^{-1}\log\|U_t^k\| < \infty$. Finally U is called *spectrally bounded* if the spectral radius of each U_t is equal to one.

Obviously each of the notions above is stronger than its successor. If U is nqa then by results of Domar [8] $L^1(G, U)$ is semisimple, regular, its Gelfand space is equal to $G^{\times}$, and the Gelfand representation is given by Fourier transformation. Moreover every proper closed ideal is contained in the kernel of some character. Thus in this case the representation of $L^1(G, U)$ by U^1 will play the same important role as the representation of $L^1(G)$ in the case where U is bounded. In the sequel we shall use all these facts without mentioning them explicitly.

The importance of non-quasianalytic representations is underlined by the following example:

3.2. Example. Let $(U_t)_{t\in\mathbf{R}}$ be a C_0-group of operators on the Banach space E. Assume that there exists $M > 0$ and $r \in \mathbf{N}$ satisfying $\|U_1^k\| \leqq M|k|^r$ for all $k \in \mathbf{Z}$. Then U is non-quasianalytic.

3.3. (S) Definition. Let U be a representation of G on E.

(a) $\chi \in G^{\times}_{\infty}$ is called an *eigenvalue* of U if there exists an $x \in E$ satisfying $0 \neq x$ and $U_t x = \chi(t)x$ for all $t \in G$. Such x is called an *eigenvector* and the linear hull of all eigenvectors corresponding to χ is the *eigenspace* $E(\chi)$. The set $\mathrm{P}\sigma(U)$ of all eigenvalues is called the *point spectrum* of U.

(b) $\chi \in G^{\times}_{\infty}$ is called an approximate eigenvalue of U if there exists a net (x_γ) (called a corresponding *approximate eigenvector*) of normalized vectors such that $(U_t x_\gamma - \chi(t)x_\gamma)$ converges to 0 uniformly on every compact subset of G as γ goes to infinity. The set $\sigma(U)$ of all approximate eigenvalues is called the *spectrum of* U.

Remarks. (1) $\chi \in \sigma(U)$ iff for all compact subsets K of G

$$0 = \inf_{\|x\|=1}\left(\sup_{t\in K}\|U_t x - \chi(t)x\|\right).$$

(2) Let U be bounded. Then our notion of $\sigma(U)$ agrees with that one of Arveson (see [6], sect. 6). More generally if U is non-quasianalytic the spectrum

can be introduced in these two equivalent ways, too ([1]). Before discussing these connections we have to prove the basic theorem of this section.

3.4. THEOREM. *Let U be a representation of G on E, and let $\tilde{U}$ be the nonstandard hull of U on $\tilde{E}_G$. Then $\sigma(U) = \mathrm{P}\,\sigma(\tilde{U}) = \sigma(\tilde{U})$.*

PROOF. For $\chi \in \sigma(U)$ arbitrary let $(x_\gamma)_{\gamma\in\Gamma}$ be a corresponding approximate eigenvector. Since χ is continuous we obtain that for γ infinitely large the following assertion holds for all $s, t \in \mathrm{ns}\,({}^*G)$ satisfying $s \approx t$:

$$U_s x_\gamma \approx \chi(s) x_\gamma \approx \chi(t) x_\gamma \approx U_t x_\gamma.$$

Hence $x_\gamma \in E_G$ and $\tilde{U}_t \hat{x}_\gamma = \chi(t)\hat{x}_\gamma$ holds for all $t \in G$. So $\sigma(U) \subset \mathrm{P}\,\sigma(\tilde{U})$.

Now assume that $\chi \notin \sigma(U)$. Then there exists a (standard) compact set K satisfying

$$\inf_{\|x\|=1} \left(\sup_{t\in K} \| U_t x - \chi(t) x \| \right) = : \delta > 0.$$

By transfer to ${}^*\mathcal{M}$ we obtain:

to each $x \in E_G$ of norm 1 there exists $t \in {}^*K$ satisfying $\| U_t x - \chi(t) x \| \geqq \delta$.

But then $\| U_s x - \chi(s) x \| \approx \| U_t x - \chi(t) x \| > \delta/2$ holds for $s = \mathrm{st}\,(t)$. Applying the quotient mapping A we obtain

$$\inf \left\{ \sup_{s\in K} (\| \tilde{U}_s \hat{x} - \chi(s)\hat{x} \|) : \hat{x} \in \tilde{E}_G,\ \| \hat{x} \| = 1 \right\} > \delta/4,$$

hence $\chi \notin \sigma(\tilde{U})$.

Let $T \in \mathcal{L}(E)$ be given. Recall that $z \in \mathbf{C}$ is called an approximate eigenvalue of T iff $\inf_{\|x\|=1} (\| Tx - zx \|) = 0$. The set of all approximate eigenvalues is denoted by $\mathrm{Ap}\,\sigma(T)$. Obviously $\mathrm{Ap}\,\sigma(T)$ is equal to the point spectrum $\mathrm{P}\,\sigma(\hat{T})$ of $\hat{T}$ (adapt the proof above). So we get easily:

3.5. COROLLARY. *Let $\chi \in \sigma(U)$ be arbitrary. Then $|\chi(t)| \leqq p(t)$ for all $t \in G$. Moreover for all $\mu \in M(G, U)$ we have $\tilde{\mu}(\sigma(U)) \subset \mathrm{P}\,\sigma(\tilde{U}^1_\mu) \subset \mathrm{Ap}\,\sigma(U^1_\mu)$.*

PROOF. By 3.4, $\chi \in \mathrm{P}\,\sigma(\tilde{U})$, hence the first assertion is obvious. But then $\int |\chi(t)|\, d\,|\mu|(t) < \infty$ for all $\mu \in M(G, U)$, and the remainder follows by an application of 2.2.

Our next aim is to establish the relationship between $\sigma(U)$ and the approximate point spectrum $\mathrm{Ap}\,\sigma(U^1)$ of U^1 defined below. In fact both spectra turn out to be equal to $\mathrm{P}\,\sigma(\tilde{U})$. The Gelfand space $\Gamma(B)$ of the algebra $L^1(G, U) =$

$: B$ is equal to $\{\chi \in G^{\times}_{\infty} : \sup\{|\chi(t)|/p(t) : t \in G\} < \infty\}$ ([9]). Similarly to the corresponding notions for U one can define eigenvalues and approximate eigenvalues of the representation U^1 of B. The set of all eigenvalues is denoted by $\mathrm{P}\,\sigma(U^1)$, that one of all approximate eigenvalues by $\mathrm{Ap}\,\sigma(U^1)$ ($\Lambda_3(U^1)$ in [9]; $\chi \in \Gamma(B)$ is in $\mathrm{Ap}\,\sigma(U^1)$ iff $\inf_{\|x\|=1}(\sup_{f\in F}\|U^1_f x - \tilde{f}(\chi)x\|) = 0$ for all finite subsets $F \subset B$).

The next proposition is not too surprising. In this general form, however, it is neither contained in [1] where only nqa representations are considered nor in [9, 18, 19] where only weaker notions of $\sigma(U)$ are treated.

3.6. Proposition. *Set* $J = \{f \in B : U^1_f = 0\}$ *and* $h(J) = \{\chi \in \Gamma(B) : \tilde{f}(\chi) = 0$ *for all* $f \in J\}$. *Then the following assertions are true*:

(a) $\sigma(U) = \mathrm{Ap}\,\sigma(U^1) = \mathrm{P}\,\sigma(\tilde{U}^1) = \mathrm{P}\,\sigma(\tilde{U}) \subset h(J)$.

(b) *For all* $f \in B$, $\tilde{f}(\sigma(U)) \subset \mathrm{Ap}\,\sigma(U^1_f) \subset \sigma(U^1_f) \subseteq \overline{\tilde{f}(h(J))}$.

Proof. (a) Similarly to 3.4 one can prove $\mathrm{Ap}\,\sigma(U^1) = \mathrm{P}\,\sigma(\hat{U}^1)$ where by definition $\hat{U}^1 : f \to (U^1_f)^\wedge \in \mathscr{L}(\hat{E})$. But now $(U^1_f)^\wedge$ map $\hat{E}$ into $\tilde{E}_G$ by 2.3. Since B is semisimple each eigenvector of $\hat{U}^1$ is contained already in $\tilde{E}_G$, hence $\mathrm{Ap}\,\sigma(U^1) = \mathrm{P}\,\sigma(\tilde{U}^1)$. Obviously $\mathrm{P}\,\sigma(\tilde{U}) \subset \mathrm{P}\,\sigma(\tilde{U}^1)$ $(\subset h(J))$. Let conversely $\chi \in \mathrm{P}\,\sigma(\tilde{U}^1)$ be arbitrary and let $\hat{x} \in \tilde{E}_G$ be a corresponding eigenvector. Choose a bounded approximate unit (f_β) in B. Then by 2.3 the following holds:

$$\tilde{U}_t\hat{x} = \lim_\beta \tilde{U}^1_{\varepsilon_t * f_\beta}\hat{x} = \lim_\beta \chi(t)\tilde{f}_\beta(\chi)\hat{x} = \chi(t)\hat{x}.$$

(Here ε_t denotes the point measure concentrated on t.)

(b) follows from 3.5 and from the fact that $h(J)$ is the Gelfand space of B/J.

3.7. Remarks. (1) $\mathrm{Ap}\,\sigma(U^1)$ is closed in $\Gamma(B)$ [9], hence $\sigma(U)$ is closed in $G^{\times}_{\infty}$.

(2) In case U is non-quasianalytic $B = L^1(G, U)$ is regular, hence $\mathrm{Ap}\,\sigma(U^1) = h(J)$ by [9], 6.5; so all sets in (a) and (b) are equal. Moreover in this case $\sigma(U)$ is never empty ([9], 6.5).

§4. Easy applications to general spectral theory

We adhere to the assumptions of section 3 unless explicitly stated otherwise.

In order to look for the relation between the spectrum of a representation $U = (U_t)_{t\in\mathbf{R}}$ and the spectrum of the corresponding infinitesimal generator A of (U_t) we want to consider the more general case of representations U of $\mathbf{R}_+ = \{t : t \geqq 0\}$ and their infinitesimal generators A.

Recall that by definition $z \in \mathrm{Ap}\,\sigma(A)$ if $\inf\{\|Ax - zx\| : \|x\| = 1, x \in D(A)\} = 0$ ($D(A)$: domain of A). By $\mathrm{P}\,\sigma(A)$ we denote the point spectrum of A.

The following result is essentially due to Derndinger [7], though he used a standard and weaker substitute for the nonstandard hull. In fact nonstandard analysis facilitates the reasoning in an important manner.

4.1. PROPOSITION. *Let U be a representation of $\mathbf{R}_+$ on E, denote by A its infinitesimal generator, and by $\tilde{A}$ the infinitesimal generator of the nonstandard hull $\tilde{U}$ on $\tilde{E}_{\mathbf{R}_+}$. Then the following assertions hold:*

(a) $\sigma(A) = \sigma(\tilde{A})$.

(b) $\mathrm{Ap}\,\sigma(A) = \mathrm{P}\,\sigma(\tilde{A})$.

PROOF. There exists $r > 0$ such that the function $f : t \to \exp(-rt)$ is in $L^1(\mathbf{R}_+, U)$ (i.e., the measure $f.dt$ is in $M(\mathbf{R}_+, U)$), and $T := (r - A)^{-1} = U_f^1$. 2.2 implies $\tilde{T} := \hat{T}_{|\tilde{E}_{\mathbf{R}_+}} = \tilde{U}_f^1 = (r - \tilde{A})^{-1}$.

Set $\sigma_0 = \sigma$, $\sigma_1 = \mathrm{P}\,\sigma$, and $\sigma_2 = \mathrm{Ap}\,\sigma$. Then the following relations are well-known for $i = 0, 1, 2 : \sigma_i(T) \setminus \{0\} \subset \{(r - z)^{-1} : z \in \sigma_i(A)\} \subset \sigma_i(T) \cup \{0\}$. They hold as well for $\tilde{A}$, $\tilde{T}$. Hence we have only to show $\sigma(T) = \sigma(\tilde{T})$ and $\sigma_2(T) \setminus \{0\} = \sigma_1(\tilde{T}) \setminus \{0\}$. But the first equation follows from 1.4(b) since $\sigma(T) = \{z \in \mathbf{C} : (z - T)$ is not invertible in $\{T\}'\}$. Like in the proof of 2.3 the integral formula $T = U_f^1$ shows $\hat{T}(\hat{E}) \subset \tilde{E}_{\mathbf{R}_+}$, hence

$$\sigma_2(T) \setminus \{0\} = \sigma_1(\hat{T}) \setminus \{0\} = \sigma_1(\tilde{T}) \setminus \{0\}.$$

The proposition above enables us to generalize a result of Evans [10] from the case of bounded representations to arbitrary ones. Again our proof is an extremely simple application of nonstandard analysis. In order to fomulate the result in a precise manner we have to identify $\mathbf{C}$ with $\mathbf{R}_\infty^\times$ by $z \to \chi_z$ ($\chi_z(t) = \exp(zt)$).

4.2. (S) PROPOSITION. *Let U be a representation of $\mathbf{R}$ on E, and let A denote its infinitesimal generator. Then* $\mathrm{Ap}\,\sigma(A) = \sigma(U)$. *If in particular U is spectrally bounded then $\sigma(A) = \sigma(U)$.*

PROOF. Let A be as in 4.2. $z \in \mathrm{P}\,\sigma(\tilde{A})$ iff there exists $0 \neq \hat{x} \in D(\tilde{A})$ satisfying $z\hat{x} = \tilde{A}\hat{x}$ iff $\chi_z \in \mathrm{P}\,\sigma(\tilde{U})$. So the first assertion follows from 3.7 and 4.1. If U is spectrally bounded then $\sigma(A) \subset i\mathbf{R}$ hence $\sigma(A) = \mathrm{Ap}\,\sigma(A)$.

4.3. (S) COROLLARY. *Let $(U_t)_{t \in \mathbf{R}}$ be a C_0-group on the Banach space $E \neq \{0\}$. If $\| U_1^k \| = O(|k|^r)$ for a suitable $r \geqq 0$ then $\sigma(A) \neq \emptyset$.*

PROOF. Use 3.2, 3.7, and 4.2.

Our next result was proved in case of a bounded representation by Olesen [23]. Moreover the assertion below that $\sigma(U) \neq \emptyset$ was shown in [9] for a weaker notion of $\sigma(U)$; again our proof is quite easy.

4.4. (S) PROPOSITION. *Let U be a uniformly continuous representation of G on E. Then $\sigma(U)$ is compact and non-void.*

Conversely if $\sigma(U)$ is compact and the representation is non-quasianalytic then it is uniformly continuous.

REMARKS. (1) Notice that in the first assertion U need not even be spectrally bounded.

(2) Nymann [22] gave a sophisticated example of a spectrally bounded representation U of $\mathbf{R}$ with void spectrum. Thus the additional hypothesis that U is nqa is not superfluous.

PROOF OF 4.4. (I) Let U be uniformly continuous. First of all we show that $\mathrm{P}\,\sigma(U)$ is relatively compact. Let $\chi \in {}^*\mathrm{P}\,\sigma(U)$ be given. Then there exists a corresponding eigenvector x of norm 1 in ${}^*E$. Since $\mathrm{fin}({}^*E) = E_G$, $U_s x \approx U_t x$ hold for all $s \approx t$ nearstandard. Thus χ is s-continuous on $\mathrm{ns}({}^*G)$, hence $\mathrm{P}\,\sigma(U)$ is relatively compact by 8.4.41 in [26]. Now by 1.6 and 3.4 we may apply the assertion established above to $\sigma(U) = \mathrm{P}\,\sigma(\hat{U})$. 3.7(1) yields that $\sigma(U)$ is compact.

(II) Let U be as before. By 6.6 in [9] there exists $\chi \in G^{\times}_{\infty}$ which is in the spectrum $\sigma(U_d)$ of the same representation U but viewed as a representation U_d of the discrete group G_d. By 3.4 there exists $x \in \mathrm{fin}({}^*E)$ of norm one such that $U_t x \approx \chi(t)x$ for all standard t. Since by 1.6, $\hat{E} = \tilde{E}_G$, we obtain $\chi \in \mathrm{P}\,\sigma(\tilde{U}) = \sigma(U)$.

(III) Let now $\sigma(U)$ be compact. By 3.7 we have $\sigma(U) = h(J)$, hence the Gelfand space of $B := L^1(G, U)/J$ is compact, which shows that B has a unit $\check{f} := f + J$. Then $U^1_f = I$ (since $L^1(G, U)$ has a bounded approximate identity), hence $\tilde{U}^1_f$ is the identity on $\hat{E}$ and 2.3 and 1.6 yield the assertion.

§5. The Riesz points of the spectrum

Let T be a bounded linear operator on the Banach space E. $z \in \mathbf{C}$ is called a Riesz point of the spectrum $\sigma(T)$ if it is a pole of the resolvent such that the residuum is of finite rank. As is known this is equivalent to the two properties that z is isolated in $\sigma(T)$ and $(z - T)$ is upper Fredholm. According to [5] this in turn holds iff z is isolated in $\sigma(T)$ and $\ker(z-\hat{T})$ is finite-dimensional. So we are led to the following notion (we adhere again to the general assumptions made in sect. 3).

5.1. (S) DEFINITION. *Let U be a representation of G on E.* $\chi \in \sigma(U)$ is called a *Riesz point* if χ is isolated in $\sigma(U)$ and if moreover every approximate eigenvector (x_γ) corresponding to χ possesses an accumulation point. The set of Riesz points is denoted by $\mathrm{R}\,\sigma(U)$, its complement in $\sigma(U)$ is called the essential spectrum $\sigma_{ess}(U)$.

Denote by B_1 the closed subalgebra of $\mathscr{L}(E)$ generated by $U^1(L^1(G,U))$ and I. Assume that U is non-quasianalytic and that χ is isolated in $\sigma(U)$. Then using 3.7, 6.5 in [9], and Shilov's idempotent theorem we get a unique projection $Q_\chi \in B_1$ which reduces U. In more details this means:

(i) $Q_\chi U_t = U_t Q_\chi$ for all $t \in G$.

(ii) For $V: t \to V_t := Q_\chi U_t$ (on $Q_\chi(E)$), $\sigma(V)$ equals $\{\chi\}$.

(iii) For $W: t \to W_t := (I - Q_\chi)U_t$ (on $(I-Q_\chi)(E)$), $\chi \notin \sigma(W)$.

If U is bounded then Q_χ maps E onto the eigenspace $E(\chi)$, so in that case $\chi \in \mathrm{R}\,\sigma(U)$ iff χ is isolated and Q_χ is of finite rank. This assertion remains true also in the more general case of a non-quasianalytic representation, as we shall show below (5.3). But first of all we need a characterization of a Riesz point in terms of nonstandard analysis:

5.2. PROPOSITION. *Let U be an arbitrary representation of G on E and let χ be isolated in $\sigma(U)$. The following assertions are equivalent:*

(a) $\chi \in \mathrm{R}\,\sigma(U)$.

(b) *The eigenspace $\tilde{E}(\chi)$ of χ in $\tilde{E}_G$ is equal to $E(\chi)$ in E.*

(c) $\dim \tilde{E}(\chi) < \infty$.

(d) $\chi \in \mathrm{R}\,\sigma(\tilde{U})$.

PROOF. (I) Set $E(\chi) = F$. Then $\mathrm{fin}(^*F)$ is contained in E_G hence $\hat{F} = \mathrm{fin}(^*F)/\mathrm{inf}(^*F)$ is a closed subspace of $\tilde{E}(\chi)$. Thus if F is infinite-dimensional then $F \neq \hat{F}$ hence $\tilde{E}(\chi) \not\subset E$.

(II) (a) $\Rightarrow$ (b): Suppose $F \neq \tilde{E}(\chi)$. Then there exists $\hat{x} \in \tilde{E}(\chi)$ of norm one satisfying $d(\hat{x}, F) := \inf\{\|\hat{x} - y\| : y \in F\} \geqq \frac{1}{2}$ (cf. [31], p. 84). Thus for $x \in \hat{x}$ of norm one we have $d(x, {}^*F) \geqq \frac{1}{4}$ and $\| U_t x - \chi(t)x \| \approx 0$ for all $t \in \mathrm{ns}(^*G)$. This particular x shows the following:

(S): If K is a compact standard set and $\varepsilon > 0$, a standard real number, then there exists an x of norm one such that $d(x, {}^*F) \geqq \frac{1}{4}$ and $\sup_{t \in {}^*K} (\| U_t x - \chi(t)x \|) < \varepsilon$.

By the transfer principle we obtain an approximate eigenvector $(x_{K,\varepsilon})_{K \text{ comp}, \varepsilon>0}$ which has no accumulation point because of $d(x_{K,\varepsilon}, F) \geqq \frac{1}{4}$.

(b) $\Rightarrow$ (c): See (I).

(c) $\Rightarrow$ (d): $F \subset \tilde{E}(\chi)$, hence $\dim F < \infty$. Suppose there exists an approximate eigenvector $(\hat{x}_\gamma)$ in $\tilde{E}_G$ (corresponding to χ) with no accumulation point. Then there is $\delta > 0$ (standard) and γ_0 (standard) such that $d(\hat{x}_\gamma, F) \geqq \delta$ for all $\gamma \geqq \gamma_0$. To every standard compact set $K \subset G$ and every standard $\varepsilon > 0$ there exists $\gamma(K, \varepsilon) \geqq \gamma_0$ such that

$$\sup_{t \in K} (\| \tilde{U}_t \hat{x} - \chi(t) \hat{x}_\gamma \|) < \varepsilon \qquad \text{for all } \gamma \geqq \gamma(K, \varepsilon).$$

Pick an $x_{\gamma(K,\varepsilon)} \in \hat{x}_{\gamma(K,\varepsilon)}$ of norm one. Since $x_{\gamma(K,\varepsilon)} \in E_G$ these elements verify assertion (S) above (with $\delta/2$ in place of $\frac{1}{4}$). Like there we obtain an approximate eigenvector (x_β) in E such that $d(x_\beta, F) \geqq \delta/2$. Thus (x_β) has no accumulation points, hence there exists β_0 and $\delta_1 > 0$ (standard) such that $\| x_\beta - x_\gamma \| \geqq \delta_1$ for all $\beta, \gamma \geqq \beta_0, \beta \neq \gamma$. But for all β infinitely large $\hat{x}_\beta$ is in $\tilde{E}(\chi)$ (cf. proof of 3.4) and $\| \hat{x}_\beta - \hat{x}_\gamma \| \geqq \delta_1/2$ for $\beta \neq \gamma$, thus $\dim \tilde{E}(\chi) = \infty$.

(d) $\Rightarrow$ (a): Obvious.

We now apply this proposition to non-quasianalytic representations.

5.3. (S) Theorem. *Let U be a non-quasianalytic representation of G on E. Then $\chi \in \mathrm{R}\,\sigma(U)$ iff χ is isolated and the reducing projection Q_χ is of finite rank.*

Proof. (I) For the nontrivial part of the assertion ($\chi \in \mathrm{R}\,\sigma(U) \Rightarrow Q_\chi$ is of finite rank) we may assume without loss of generality that $Q_\chi = I$, i.e., $\sigma(U) = \{\chi\}$ and U is uniformly continuous. By 5.2, $E(\chi)$ is finite-dimensional, so there exists a continuous projection P onto $E(\chi)$. Let S be equal to $I - P$ and set $M = S(E)$. Then obviously $\hat{S}(\hat{E})$ is (identifiable with) $\hat{M}$.

(II) *Claim*: there exists a finite subset H of $L^1(G, U)$ such that

$$\inf_{x \in M, \|x\|=1} (\sup\{\| U_f^1 x - \tilde{f}(\chi) x \| : f \in H\}) = \delta > 0.$$

For else $\hat{M} \cap \tilde{E}(\chi) \neq \{0\}$, a contradiction to 5.2. Let H be $\{f_1, \dots, f_r\}$ and set $U_{f_j}^1 := T_j$. By 3.7, $\sigma(T_j) = \{\tilde{f}_j(\chi)\}$, hence the joint spectrum $\sigma(T_1, \dots, T_r)$ equals $\{z\}$ where $z = (\tilde{f}_1(\chi), \dots, \tilde{f}_r(\chi))$.

(III) (II) shows that $T : E \to E^r$ given by $Tx = (T_j x - \tilde{f}_j(\chi) x)_{j=1,\dots,r}$ is an upper Fredholm operator. Thus the upper Fredholm spectrum of $(T_1, \dots, T_r)$ is empty hence $\dim E < \infty$ by [4], theorem 7.

The following result may be viewed as a spectral mapping theorem for the Riesz part of the spectrum.

5.4. (S) PROPOSITION. *Let U be an arbitrary representation of G on E. Assume that there exists $\mu \in M(G, U)$ such that $\sigma(U^1_\mu)$ contains a Riesz point z. Then $D := \{\chi \in \sigma(U) : \tilde{\mu}(\chi) = z\}$ is nonempty, finite, and is contained in* $\mathrm{R}\,\sigma(U)$.

PROOF. $H := \ker(z - U^1_\mu) = \ker(z - \hat{U}^1_\mu)$ is finite-dimensional (cf. the beginning of this section). Thus $t \to V_t := U_{t|H}$ is uniformly continuous. By 4.4 there exists $\chi \in \sigma(V)\,(\subset \sigma(U))$. 3.5 yields $z = \tilde{\mu}(\chi)$ hence $D \neq \emptyset$. But by 2.2 we have $\tilde{E}(\eta) \subset H$ for all $\eta \in D$. $\dim H < \infty$ implies that D is finite. Finally z is isolated in $\sigma(U^1_\mu)$ and by 3.5, $D = \tilde{\mu}^{-1}(z)$, hence D is clopen in $\sigma(U)$ and the assertion follows from 5.2.

We now turn to the characterization of those representations U satisfying $\sigma(U) = \mathrm{R}\,\sigma(U)$.

5.5. (S) DEFINITION. The representation U is called R-compact if every bounded subset C of E satisfying $\lim_{t \to e}(\sup\{\| U_t x - x \| : x \in C\}) = 0$ is relatively compact.

R-compact representations are in a certain sense opposite to uniformly continuous ones. They are characterized in the following theorem:

5.6. (S) PROPOSITION. *For a representation U of G on E the following two assertions are equivalent*:

(a) *U is R-compact.*

(b) *U^1_f is compact for all $f \in L^1(G, U)$.*

Moreover any of these two assertions imply

(c) $\sigma(U) = \mathrm{R}\,\sigma(U)$.

We formulate the nonstandard part of the theorem separately:

5.7. PROPOSITION. *Any of the assertions* (a) *or* (b) *above is equivalent to the following one*:

(d) $\tilde{E}_G = E$.

PROOF. (a) $\Rightarrow$ (b): Since $t \to \varepsilon_t * f$ is continuous from G into $L^1(G, U)$ and since U^1 is a contraction it is easy to see that (U_t) converges to I uniformly on $\{U^1_f x : \| x \| = 1\}$ for every $f \in L^1(G, U)$. By (a), U^1_f is compact.

(b) $\Rightarrow$ (d): $T \in \mathscr{L}(E)$ is compact iff $\hat{T}$ maps $\hat{E}$ into E (cf. [20]). Hence (d) follows from 2.3.

(d) $\Rightarrow$ (a): Let $C \subset E$ be bounded and assume that

$$\lim_{t\to e}\left(\sup_{y\in C}\|U_t y - y\|\right) = 0.$$

Then for $y \in {}^*C$ and $t \approx e$, $U_t y \approx y$ hence ${}^*C \subset E_G = \mathrm{ns}({}^*E)$ by assumption; so C is relatively compact by [26], 8.3.1.

Finally assume (b). To $\chi \in \sigma(U)$ there exists $f \in L^1(G, U)$ satisfying $\hat{f}(\chi) \neq 0$. Now apply 5.4.

5.8. (S) Corollary. *Let U be non-quasianalytic. Assume that there is an $f \in L^1(G, U)$ vanishing nowhere on $\sigma(U)$ such that U_f^1 is compact. Then U is R-compact.*

Proof. If f vanishes nowhere on $\sigma(U)$ then U_f^1 has dense range. This may be proved along the lines of the proof of the corresponding result for bounded representations in [3], sect. 1. But since $\sigma(U) = \sigma(\tilde{U})$, $\tilde{U}_f^1(\tilde{E}_G)$ is dense in $\tilde{E}_G$. Since U_f^1 is compact we have $\tilde{U}_f^1(\tilde{E}_G) \subset \hat{U}_f^1(\hat{E}) \subset E$, hence 5.7(d) applies.

In order to formulate the final main result of this section we recall that $T \in \mathcal{L}(E)$ is called a Riesz operator if $\sigma(T)\setminus\{0\}$ consists of Riesz points only. The following theorem gives a converse to 5.6 (a) $\Rightarrow$ (c) in case of a non-quasianalytic representation.

5.9. (S) Theorem. *Let U be a non-quasianalytic representation of G on E. The following assertions are equivalent*:

(a) $\sigma(U) = \mathrm{R}\,\sigma(U)$.

(b) *U is R-compact.*

(c) *For all $f \in L^1(G, U)$, U_f^1 is a Riesz operator.*

(d) *For all $f \in L^1(G, U)$, U_f^1 is compact.*

Proof. In view of 5.6 we only have to prove (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): The spectral projections Q_χ ($\chi \in \sigma(U)$) are all of finite rank by 5.3. Thus $\hat{Q}_\chi$ maps $\hat{E}$ into E for every χ. Let F be the closed linear span of $\bigcup_{\chi\in\sigma(U)} \hat{Q}_\chi(\hat{E})$. Then F ($\subset \tilde{E}_G$) is invariant under $U(G)$. Consider the induced representation S on the quotient $\tilde{E}_G/F$. Then on one hand we have $\sigma(S) \subset \sigma(U) = \mathrm{R}\,\sigma(U)$ by 5.2. But since $\hat{Q}_\chi(\hat{E}) = \hat{Q}_\chi(E) \subset F$ we obtain $\chi \notin \sigma(S)$, so $\sigma(S) = \varnothing$ and hence $E \supset F = \tilde{E}_G$ by 3.7. Thus (b) follows from 5.7.

(c) $\Rightarrow$ (a): This follows from 5.4 and the regularity of $L^1(G, U)$.

5.10. (S) Corollary. *Let U be non-quasianalytic. Assume that there exists $\mu \in M(G, U)$ such that μ vanishes nowhere on $\sigma(U)$ and that in addition U_μ^1 is a*

Riesz operator. Then $\sigma(U) = \mathrm{R}\,\sigma(U)$, *hence all the other assertions of* 5.9 *are true.*

PROOF. By 3.6, $\tilde{\mu}(\sigma(U)) \subset \sigma(U^1_\mu)$, hence 5.4 yields $\sigma(U) = \mathrm{R}\,\sigma(U)$.

§6. Applications to representations on Banach lattices

In this section [25] serves as a general reference concerning Banach lattices. We adhere to the assumptions made at the beginning of section 3. Moreover we always assume that E is a Banach lattice (over $\mathbf{C}$), and that each U_t is a positive operator (in fact then each U_t is a lattice isomorphism) In this case we call U a *lattice action.* Unless otherwise stated explicitly we consider only lattice actions in this section. If χ is an unbounded character on G then $\chi = |\chi| \cdot \eta$ where $|\chi|$ is a realvalued character and η is a bounded one.

6.1. (*S*) PROPOSITION. *Let* U *be a lattice action of* G *on* E. *Then* $\sigma(U)$ *is cyclic, i.e., if* $\chi = |\chi|\eta \in \sigma(U)$ *then* $\{|\chi|\eta^n : n \in \mathbf{Z}\} \subset \sigma(U)$. *If in particular* U *is non-quasianalytic then* $1_G \in \sigma(U)$.

PROOF. In view of 1.4 and 3.4 we can assume that $\chi \in \mathrm{P}\,\sigma(U)$, i.e., there exists a nonzero $x \in E$ such that $U_t x = \chi(t)x$ holds for all t, hence $U_t u = |\chi(t)|\, u$ for $u = |x|$. Therefore the closure F of the principal ideal $E_u = \bigcup_{n\in\mathbf{N}} \{z \in E : |z| \leqq nu\}$ is invariant under $U(G)$ and contains x. Consider

$$V_t = \frac{1}{|\chi(t)|} U_{t|F}.$$

We have $V_t u = u$, $V_t x = \eta(t)x$, and now we proceed in complete analogy to [25], proof of cor. 2 on p. 324, in order to obtain the result (cf. [28], too). The last assertion follows from 3.7.

6.2. (*S*) COROLLARY. *Let* U *be a spectrally bounded lattice action of* $\mathbf{R}$ *on* E, *and let* A *be the infinitesimal generator of* $U(\mathbf{R})$. *Then either* 0 *is isolated in* $\sigma(A)$ *or* $\sigma(A) = i\mathbf{R}$.

PROOF. By [15], $\sigma(A) \neq \emptyset$, hence $0 \in \sigma(A)$ by 6.1 and 4.3. If 0 is not isolated in A then $\sigma(A) = i\mathbf{R}$ by 6.1, 4.2, and by the fact that $\sigma(U) = \sigma(A)$ is closed.

Recall the following notions: a lattice action U on E is called irreducible if there is no nontrivial closed $U(G)$-invariant lattice ideal. By definition U is non-degenerate [28] if to every pair of disjoint compact sets $K, L \subset G$ there exists $x > 0$ in E such that $\inf(U_s x, U_t x) = 0$ for all $(s, t) \in K \times L$. A positive

$u \in E$ is called a topological order unit of E if $\{z : |z| \leqq u\}$ is total in E, or equivalently, if the ideal $E_u = \bigcup_n \{z : |z| \leqq nu\}$ is dense in E.

As an application of sect. 5 we obtain

6.3. (S) THEOREM. *Let U be a spectrally bounded lattice action of G on the Banach lattice E. Assume that there exists a measure $\mu \in M(G, U)$ such that the spectrum of U^1_μ contains a Riesz point.*

The following assertions are true:

(a) *If U is bounded and irreducible then $H := G/\mathrm{Ker}\ U$ is compact and U is isomorphic (in an obvious sense) to the canonical action V of G on a space F of functions f on H $((V_t f)(s\ \mathrm{Ker}\ U) = f(ts\ \mathrm{Ker}\ U))$ where $C(H) \subset F \subset L^1(H)$.*

(b) *If G is discrete and U is non-degenerate then G is finite.*

(c) *If U is non-degenerate, non-quasianalytic and if $U(G)$ possesses a fixed point which is a topological order unit then G is already compact.*

PROOF. (a) By 5.4 there exists $\chi \in \mathrm{R}\,\sigma(U)$. Let $0 \neq x \in E(\chi)$ be arbitrary. Then $U_t|x| = |U_t x| = |\chi(t)|\,|x| = |x|$. Since U is irreducible $U = |x|$ is a topological order unit and the assertion follows from [28], 4.7. (Concerning the isomorphism cf. the Halmos–von Neumann theorem [21].)

(b) By [30], $\sigma(U) = G^x$. 5.4 implies that the compact group G^x contains an isolated point. Hence G is finite.

(c) By [28], $\sigma(U) = G^x$. 5.4 implies therefore that G^x is discrete, hence G is compact.

6.4. (S) COROLLARY. *If U satisfies any of the assumptions made in* (a), (b) *or* (c), *and if in addition $U(G)$ is not compact then for every $\mu \in M(G, U)$ the spectrum $\sigma(U^1_\mu)$ of U^1_μ is equal to its essential part $\sigma_{ess}(U^1_\mu)$.*

We finally give a characterization of R-compact bounded representations. Our result generalizes [12], 4.5, and [27], obtained there for $G = \mathbf{R}$. In fact we show that the apparently different assumptions made in these two papers are in fact equivalent.

6.5. (S) DEFINITION. A lattice action U of G on E is called compactly reducible if there exist finitely many mutually disjoint bands $E_1, \ldots, E_r$ with the following properties:

(i) $E = E_1 + \cdots + E_r$.

(ii) For every $j = 1, \ldots, r$, E_j is invariant under $U(G)$, the restriction $U_j : t \to U_{t|E_j}$ is irreducible on E_j and $G/\mathrm{Ker}\ U_j$ $(= U_j(G))$ is compact.

If U is compactly reducible, U is conjugate (in an obvious sense) to the action of G on the direct sum of appropriate function spaces F_j on $G/\mathrm{Ker}\ U_j =: H_j$

where on F_j the action is induced by that one of G on H_j (cf. the abstract Halmos–von Neuman theorem [21], or [25], III. 10.4).

6.6. (S) THEOREM. *Let U be a bounded lattice action of G on E. The following assertions are equivalent*:

(a) *There exists a positive bounded measure μ on G such that the spectral radius $r(U^1_\mu)$ of U^1_μ is a Riesz point in $\sigma(U^1_\mu)$.*

(b) *U is compactly reducible.*

(c) *For all $f \in L^1(G)$ U^1_f is compact.*

(d) *U is R-compact.*

REMARK. If $G = \mathbf{R}$ then one may consider $\mu = f.dt$ where

$$f(t) = \begin{cases} \exp(-t) & t \geqq 0, \\ 0 & t < 0. \end{cases}$$

Then $U^1_\mu = (1-A)^{-1}$ where A is the infinitesimal generator of $U(\mathbf{R})$. Thus in this case Greiner's result [12], 4.5 corresponds to the equivalence of (a) and (b) above, and Uhlig's result [27] to the equivalence of (a) and (c).

PROOF. In view of 5.9 it remains to prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b): By 5.4, $1_G \in \mathrm{R}\,\sigma(U)$. Then it is not hard to see that $E(1_G)$ is even a sublattice, hence isomorphic to $\mathbf{C}^r$ ([25], p. 70) ($r = \dim E(1_G)$). Thus there exists a basis $x_1, \ldots, x_r$ of positive mutually orthogonal elements.

Claim: $u = \Sigma x_j$ is a topological order unit. For let F denote $\overline{\bigcup_{n \geq 1}\{z : |z| \leqq nu\}}$, and let P be the spectral projection corresponding to 1_G (see sect. 5). Then $P(E) = E(1_G) \subset F$. F is obviously $U(G)$-invariant. Hence there is an induced bounded lattice action S of G on the quotient space E/F. 3.7 and 6.1 together imply $1_G \in \sigma(S)$ if $E/F \neq \{0\}$. On the other hand the associated spectral projection, Q, say, has to be zero. So $F = E$, i.e., u is a topological order unit. Since all x_j are fixed points of $U(G)$ an induction argument shows $E = \bigoplus_{j=1}^r E_j$ where $E_j = x_j^{\perp\perp}$ is the band generated by x_j which is obviously $U(G)$-invariant.

Claim: U_j is irreducible. Without loss of generality assume $E_j = E$, in particular $\dim E(1_G) = 1$; set $x_j = u$. Now if $J \neq \{0\}$ is $U(G)$-invariant, then by 3.7 and 6.1, $1_G \in \sigma(V)$ where $V : t \to U_{t|J}$, hence $u \in J$. Like above we can easily prove that $\bar{E}_u = E$, hence $J = E$. Thus 6.3(a) yields that $G/\mathrm{Ker}\, U$ is compact.

(b) $\Rightarrow$ (c): Without loss of generality assume that $E = E_1$ and Ker $U = \{e\}$. By [28], 4.7, $\sigma(U) = G^x$, in particular $\sigma(U)$ is discrete. Since U is bounded for each $\chi \in G^x$ the spectral projection Q_χ maps E onto the eigenspace $E(\chi)$ which is one-dimensional (apply the Halmos–von Neumann theorem [21], [25], 10.4). Thus the spectral projection $\tilde{Q}_\chi$ corresponding to $\tilde{U}$ maps $\tilde{E}_G$ onto $\tilde{E}(\chi) = E(\chi)$. Since G is compact, the union of the eigenspaces of U is total in E (see [11]). But the same is true for $\tilde{U}$, hence $E = \tilde{E}_G$, and 5.7 applies.

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